7. Kruzhkov, S.N., Methods of constructing a generalized solution of the Cauchy problem for a first order quasi-linear equation. Uspekhi matem. nauk, Vol. 20, №6, 1965.
8. Sushko, V.G., On the error of approximate solutions of the Cauchy problem for a first order quasi-linear equation. Matem. zametki, Vol. 8, №3, 1970.
9. Gel'fand, I.M., Some problems on the theory of quasi-linear equations. Uspekhi matem. nauk, Vol. 14, №2, 1959.
10. Bykhovskii, E.B., On self-similar solutions of a type of propagating wave of a quasi-linear equation and of a system of equations which describe the flow of water in a sloping channel. PMM Vol. 30, №2, 1967.
11. Natanson, I.P., Theory of the Functions of Real Variable. M., Gostekhizdat, 1959.
12. Vorob'ev, A.P., On periods of solutions in the case of a center. Dokl. Akad. Nauk, BSSR, Vol. 6, №5, 1962.
13. Pontriagin, L.S., Asymptotic behavior of solutions of the systems of differential equations with a small parameter at higher derivatives. Izv. Akad. Nauk SSSR, Ser. matem, Vol. 21, №5, 1957.
14. Mishchenko, E.F., Asymptotic theory of relaxation oscillations described by second order systems. Matem. sb. Vol. 44 (86), №4, 1958.

Translated by L.K.

ON CONVECTION ONSET IN A SELF-GRAVITATING FLUID SPHERE WITH INTERNAL HEATING

PMM Vol. 35, №6, pp. 1000-1014, 1971
V.G. BABSKII and I.L. SKLOVSKAIA
(Kharkov)
(Received July 9, 1970)

Proof is given that the loss of stability of the equilibrium state of a self-gravitating fluid filling a rigid sphere having uniformly distributed internal heat sources is accompanied by the onset of a stationary axisymmetric (correct to within an arbitrary rotation) flow which remains stable in the vicinity of the point of stability loss. This flow is numerically defined as a segment of the Liapunov-Schmidt series. The problem of thermal instability of a self-gravitating fluid sphere is associated with various theories and hypotheses of geo- and astro-physics, as well as with the study of fluid behavior in conditions of quasi-weightlessness. Earlier investigations were mainly directed toward the formulation and solution of the linearized problem and finding the limit of instability [1]. Their results were further developed in later publications([2, 3] and others). The method proposed by Chandrasekhar [1] was applied in [4] to the related nonlinear problem. The theory of solution branching of equations of stationary convection [5, 6] is applied below to the study of convection onset in a self-gravitating fluid sphere.

1. Statement of problem. A rigid sphere S or radius a is filled with a viscous incompressible fluid acted upon by a spherically symmetric radial gravitational field $^4/_3\pi\rho G\mathbf{r} \equiv g\mathbf{r}$ (G is the gravitational constant and ρ the fluid density). Heat sources of constant intensity τ are distributed in the fluid which is stationary (the condition of mechanical equilibrium is satisfied [7]) and whose temperature and pressure depend only on the radius. The wall temperature is constant ($T|_S = \text{const}$).

The loss of stability of the fluid equilibrium state may lead to the onset of free convection. The "principle of stability fluctuation" [1] is satisfied in this problem, hence the least eigenvalue R of the linearized problem of velocity, pressure and temperature perturbations defines the limit of instability

$$\Delta\mathbf{u} = \nabla p - R\theta\mathbf{r},\ \nabla\mathbf{u} = 0,\ \Delta\theta = -ru_r,\ \mathbf{u}|_S = 0,\ \theta|_S = 0 \tag{1.1}$$

The generated convection motion is described by the solution of equations of stationary free convection [1, 7]

$$\Delta\mathbf{u} = (\mathbf{u}\nabla)\mathbf{u} + \nabla p - R\theta\mathbf{r},\ \nabla\mathbf{u} = 0,\ \mathbf{u}|_S = 0$$
$$\Delta\theta = P\mathbf{u}\nabla\theta - ru_r,\ \theta|_S = 0\ \left(R = \frac{g\beta\tau a^6}{3\chi^2\nu},\ P = \frac{\nu}{\chi}\right) \tag{1.2}$$

where R and P are, respectively, the Rayleigh and the Prandtl numbers, and ν, χ and β are the coefficients of kinematic viscosity, thermal diffusivity, and linear expansion of the fluid, respectively.

2. The operator equations. Let H be the space of pairs $z = (\mathbf{u}, \theta)$ and $(\mathbf{u} \in H_1,\ \theta \in H_2)$ with norm

$$\|z\|_H = (\|\mathbf{u}\|^2_{H_1} + \|\theta\|^2_{H_2})^{1/2} \tag{2.1}$$

where H_1 and H_2 are Hilbert spaces introduced in [8].

Lemma 2.1. Problem (1.2) is equivalent to the operator equation

$$z = B(R,\ z) \qquad (z \in H) \tag{2.2}$$

with the absolutely continuous operator B, while problem (1.1) is equivalent to the operator equation

$$z = A(R),\ z \tag{2.3}$$

with the absolutely continuous operator A which is the Fréchet differential of operator $B(R,\ z)$ at point $z = 0$.

Proof. Problems (1.1) and (1.2) reduce to the operator equations (see [8])

$$\mathbf{u} = -K_1(\mathbf{u}\nabla)\mathbf{u} + RK_1(\theta\mathbf{r}), \qquad \theta = -PL_1(\mathbf{u}\nabla\theta) + L_1(ru_r) \tag{2.4}$$

$$\mathbf{u} = RK_1(\theta\mathbf{r}), \qquad \theta = L_1(ru_r) \qquad (\mathbf{u} \in H_1,\ \theta \in H_2) \tag{2.5}$$

Operators K_1 and L_1 have been defined in [8], where it is also shown that K_1 and K_2 ($K_2u = -K_1(\mathbf{u}\nabla)\mathbf{u}$) are absolutely continuous in H_1, while L_1 is absolutely continuous in H_2. We shall prove that operator L_2 ($L_2z = -PL_1(\mathbf{u}\nabla\theta)$) acting from H into H_2 is absolutely continuous. To do this it is sufficient to show that this operator transforms any sequence $\{z^m\}$ weakly convergent in H into a sequence $\{L_2z^m\}$ strongly convergent in H_2. Using the integral identity

$$\int_\Omega \nabla\theta\mathbf{u}\varphi\, dx = -\int_\Omega \theta\mathbf{u}\nabla\varphi\, dx \qquad (\theta,\ \varphi \in H_2,\ \mathbf{u} \in H_1) \tag{2.6}$$

together with the Holder inequality and the imbedding theorem, we obtain

$$(L_2z^m - L_2z^n,\ \varphi)_{H_2} \leqslant C_1(\|\mathbf{u}^m - \mathbf{u}^n\|_{L_4}\|\theta^n\|_{H_2} + \|\theta^m - \theta^n\|_{L_4}\|\mathbf{u}^n\|_{H_1})\|\varphi\|_{H_2} \tag{2.7}$$

Since sequencies $\{\mathbf{u}^m\}$ and $\{\theta^m\}$ are strongly convergent in L_4, hence from inequality (2.7) follows that $\{L_2 z^m\}$ is strongly convergent in H_2, if in the latter we assume that $\varphi = L_2 z^m - L_2 z^n$.

We introduce the following notation:

$$A(R)z = (RK_1(\theta\mathbf{r}),\ L_1(ru_r)) \quad (z = (\mathbf{u}, \theta),\ \mathbf{u} \in H_1,\ \theta \in H_2)$$
$$D(z, z) = (-K_1(\mathbf{u}\nabla)\mathbf{u}, \quad -PL_1(\mathbf{u}\nabla\theta)) \tag{2.8}$$
$$B(R, z) = A(R)z + D(z, z)$$

Operators A, B and D are absolutely continuous in H. The last statement of the Lemma follows from the estimates

$$\|K_2\mathbf{u}\|_{H_1} \leqslant C_2\|z\|_H^2, \qquad \|L_2 z\|_{H_2} \leqslant C_3\|z\|_H^2 \qquad (z \in H) \tag{2.9}$$

the first of which has been proved in [9] and the second from an inequality similar to (2.7). The Lemma is proved.

Let R_0 be the eigenvalue of problem (1.1) and $\zeta = (\mathbf{u}, \theta)$ its related eigenvector. Then the corresponding eigenvector of the conjugate problem

$$\Delta\mathbf{w} = \nabla q - \tau\mathbf{r},\ \nabla\mathbf{w} = 0,\ \Delta\tau = -R_0 r w_r,\ \mathbf{w}\,|\,s = 0,\ \tau\,|\,s = 0 \tag{2.10}$$

is of the form

$$\eta = (\mathbf{w}, \tau), \quad \mathbf{w} = R_0^{-1}\mathbf{u}, \quad \tau = \theta \tag{2.11}$$

From (2.10) also follows that

$$A^*(R)z = (K_1(\tau\mathbf{r}),\ RL_1(rw_r))\ (z = (\mathbf{w}, \tau),\ \mathbf{w} \in H_1,\ \tau \in H_2) \tag{2.12}$$

3. The spectral problem. The linearized system of Eqs. (1.1) permits the separation of variables. when seeking the solution in the form of series in generalized spherical functions $T^l_{mn}(^1/_2\pi - \varphi, \vartheta, 0)$ [10]. As the result, we obtain a problem of eigenvalues with respect to parameter R for the system of ordinary differential equations [11]

$$D_l v_l = \frac{2}{r^2}u_l - \frac{1}{r}p_l \quad \left(D_l = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2}\right),\ v_l(1) = 0$$
$$D_l u_l = \frac{2}{r^2}u_l + \frac{2l(l+1)}{r^2}v_l + \frac{dp_l}{dr} - Rr\theta_l,\ u_l(1) = 0 \tag{3.1}$$
$$\frac{du_l}{dr} + \frac{2}{r}u_l + \frac{l(l+1)}{r}v_l = 0,\ D_l\theta_l = -ru_l,\ \theta_l(1) = 0\ (l = 1, 2, 3, \ldots)$$

with conditions of boundedness in zero. We eliminate p_l and v_l from system (3.1)

$$D_l^2\omega_l = Rl(l+1)\theta_l \quad (\omega_l = ru_l), \quad \omega_l(0) = \omega_l(1) = \omega_l'(1) = 0, \quad \omega_l'(0) < \infty$$
$$D_l\theta_l = -\omega_l, \quad \theta_l(1) = 0, \quad \theta_l(0) < \infty \tag{3.2}$$

Green's function $G_{1l}(r, s)$ of operator $-r^2D_l$ with boundary conditions $u(1) = 0$ and $u(0) < \infty$ and Green's function $G_{2l}(x, s)$ of operator $r^2D_l^2$ with boundary conditions $u(1) = u'(1) = u(0) = 0$ and $u'(0) < \infty$ are symmetric and of the form

$$G_{1l}(r, s) = \frac{r^l(s^{-l-1} - s^l)}{2l+1},\ r \leqslant s,\ G_{2l}(x, s) = \frac{\eta_1(x)\psi_1(s) + \eta_2(x)\psi_2(s)}{4(2l-1)(2l+1)(2l+3)},\ x \leqslant s \tag{3.3}$$
$$\eta_1(x) = (l+2)x^l - lx^{l+2}, \qquad \eta_2(x) = x^l - x^{l+2}$$

$$\psi_1(s) = (2l-1)s^{l+2} - (2l+3)s^l + (2l+3)s^{-l+1} - (2l-1)s^{-l-1}$$
$$\psi_2(s) = (l+2)(2l-1)s^{l+2} - (l-1)(2l+3)s^l - \tag{3.4}$$
$$- l(2l+3)s^{-l+1} + (l+2)(2l-1)s^{-l-1}$$

We define the integral operators G_{1l} and G_{2l} by formulas

$$G_{kl}\psi = \int_0^1 G_{kl}(x, y)\psi(y)y^2dy \qquad (k = 1, 2;\ l = 1, 2, \ldots) \tag{3.5}$$

With the use of these operators the problem (3.2) reduces to the integral equation

$$\theta_l = \lambda_l G_l \theta_l \qquad (\lambda_l = Rl(l+1),\ G_l = G_{1l}G_{2l}) \tag{3.6}$$

Lemma 3.1. Operator G_l is oscillatory.

The conditions for the one-pair kernels [12] to be oscillatory are satisfied for $G_{1l}(r, s)$ and the oscillatory character of $G_{2l}(x, s)$ is implied by the admissibility of presenting operator $r^2D_l^2$ in the form

$$r^2D_l^2u = r^{-l}\frac{d}{dr}r^{2l+2}\frac{d}{dr}r^{-2l}\frac{d}{dr}r^{2l+2}\frac{d}{dr}r^{-l}u \tag{3.7}$$

and by the results presented in [12] (see [13], Lemma 3.2). The kernel of operator G_l is oscillatory, since it is a composition of oscillatory kernels [14].

It follows from Lemma 3.1 that for any l $(l = 1, 2, \ldots)$ operator G_l has a sequence of simple positive characteristic values $0 < \lambda_{1l} < \lambda_{2l} < \ldots$, which implies that

$$0 < R_{1l} < R_{2l} < \ldots \tag{3.8}$$

The sought minimum characteristic value of operator $A(R)$ is $\min_l R_{1l}(l = 1, 2, \ldots)$.

Let K be the cone of nonnegative functions from $C[0, 1]$. We set $u_0 = r^l(1-r)$. and separate in the space $C[0, 1]$ the subspace E_{u_0} of functions with the finite norm u_0. By definition [15]

$$u \in E_{u_0}: -\alpha_1 u_0 \leqslant u \leqslant \alpha_2 u_0;\ \ \alpha_1, \alpha_2 > 0;\ \ \|u\|_{u_0} = \max\{\inf\alpha_1, \inf\alpha_2\} \tag{3.9}$$

where inf is taken over all α_1 and α_2 for which these inequalities are satisfied. We introduce the subsidiary cone $K_{u_0} = E_{u_0} \cap K$ which is normal to and solid in E_{u_0} [15].

Lemma 3.2. Operator G_l is strongly positive with respect to cone K_{u_0}.

Proof. The Lemma implies that for any arbitrary function $u \in K_{u_0}$ function $G_l u$ is an internal element of cone K_{u_0}. We note that, owing to the oscillatory properties of the kernel of $G_{2l}(x, s)$ (Lemma 3.1), cone K is invariant with respect to operator G_{2l}. Hence it is sufficient to prove that operator G_{1l} transform cone K into an inner part of cone K_{u_0}.

Green's function $G_{1l}(r, s)$ satisfies condition $G_{1l}(r, s) > 0$ $(0 < r, s < 1)$, as well as the following readily verified conditions:

$$\left.\frac{\partial^k G_{1l}(r, s)}{\partial r^k}\right|_{r=0} = 0, \quad \left.\frac{\partial^l G_{1l}(r, s)}{\partial r^l}\right|_{r=0} > 0, \quad G_{1l}(1, s) = 0, \quad \left.\frac{\partial G_{1l}(r, s)}{\partial r}\right|_{r=1} < 0$$
$$(k = 1, 2, \ldots, l-1)$$

Thus operator G_{1l} transforms any function $u \in K$ into a function of the form

$$r^l(1-r)f(r), \quad f(r) > 0 \quad (0 \leqslant r \leqslant 1) \tag{3.10}$$

From (3.10) follow the inequalities

$$\alpha u_0 \leqslant G_{1l} u \leqslant \beta u_0, \quad \alpha = \min f(r), \quad \beta = \max f(r) \quad (0 \leqslant r \leqslant 1) \tag{3.11}$$

Since u_0 is an internal element of cone K_{u_0} [15], it follows from the inequalities (3.11) that G_{1l} is also an internal element of K_{u_0}. The Lemma is proved.

Corollary. The eigenfunction $\theta_{1l}(r)$ corresponding to the characteristic value R_{1l} is an internal element of cone K_{u_0} , i.e., it is of the form (3.10).

Lemma 3.3. Operator $l(l+1)G_l$ is monotonic with respect to parameter l, $lG_l > (l+2)G_{l+1}$.

Proof. Let us prove the monotonousness of operator G_{1l}

$$G_{1l} > G_{1,\,l+1} \tag{3.12}$$

Indeed, for $r \leqslant s$

$$G_{1,\,l+1}(r,s) - G_{1l}(r,s) = z^l s^{-1} f(z,s) \qquad (z = rs^{-1} \leqslant 1)$$

$$f(z,s) = -\frac{1}{2l+3} z s^{2l+3} + \frac{1}{2l+1} s^{2l+1} - \left(\frac{1}{2l+1} - \frac{1}{2l+3} z\right)$$

Since $f(z,1) = 0$ and $\partial f / \partial s > 0$ $(0 < s < 1)$, function $f(z,s) < 0$ when $0 < r \leqslant \leqslant s < 1$. By virtue of the symmetry of Green's function $G_{1l}(r,s)$ the inequality $G_{1,l+1}(r,s) < < G_{1l}(r,s)$ holds also for $1 > r \geqslant s > 0$.

Let us now prove that

$$lG_{2l} > (l+2)G_{2,\,l+1} \tag{3.13}$$

Indeen for $x \leqslant s$

$$(l+2)G_{2,\,l+1}(x,s) - lG_{2l}(x,s) = \frac{x^l s^{-l-2}}{4(2l+1)(2l+3)} P(x,s)$$

$$P(x,s) = -a_1(s)x^3 + a_2(s)x^2 + a_3(s)x - a_4(s)$$

$$(a_i(s) > 0 \quad \text{for} \quad 0 < s < 1)$$

Function $P(x,s) < 0$ $(0 < x \leqslant s < 1)$. Owing to the symmetry of Green's function $G_{2l}(x,s)$ we have $(l+2)G_{2,l+1}(x,s) < l\,G_{2l}(x,s)$ also for $1 > x \geqslant s > 0$. Since $G_l = = G_{1l}G_{2l}$ the proof of this Lemma follows from (3.12) and (3.13).

The positiveness and monotonousness of operator $l(l+1)G_l$ implies the monotonousness of its minimum characteristic value $R_{1l} < R_{1,l+1}$. Hence it follows from inequality (3.8) that R_{11} is the minimum characteristic value of operator $A(R)$. The dimensionality of the eigen-subregion corresponding to that value is equal three, and there are no adjoint vectors. This can readily be proved by, for example, symmetrizing operator A or simply by using Lemma 1.5 in [9]. Let us prove that $(\zeta,\eta)_H$ where $\zeta = = (\mathbf{u},\theta)$ is the eigenvector of operator $A(R_{11})$, and $\eta = (\mathbf{w},\tau)$ is that of operator $A^*(R_{11})$ (see (2.11)), is not zero

$$(\zeta,\eta)_H = (\mathbf{u},\mathbf{w})_{H_1} + (\theta,\tau)_{H_2} = R_{11}^{-1}\|\mathbf{u}\|_{H_1}^2 + \|\theta\|_{H_2}^2 > 0$$

4. The bifurcation point and branching. As shown in Sect. 3, operator $A(R)$ has R_{11} as its odd-multiple minimum value. (We shall denote it by R_*). This together with Lemma 2.1 make it possible to apply to Eq. (2.2) Krasnosel'skii's theorem on points of bifurcation, as modified in [16].

Theorem 4.1. Point $R_* = R_{11}$ is a point of bifurcation for Eq. (2.2).

Thus branching of solutions of Eqs. (1.3) occurs when the Rayleigh number equals R_*. Let us examine the branching at this point by the method of Liapunov-Schmidt.

We select in the eigen-subspace Z_0 of operator $A\ (R_*)$ the following basis:

$$\zeta_k = (u(r)\, T^1_{0k}(^1/_2\pi - \varphi, \vartheta, 0),\ v(r)\, T^1_{1k}(^1/_2\pi - \varphi, \vartheta, 0),\ v(r)\, T^1_{-1k}(^1/_2\pi - \varphi, \vartheta, 0)$$

$$\theta(r)\, T^1_{0k}(^1/_2\pi - \varphi, \vartheta, 0)) \qquad (k = -1, 0, 1) \tag{4.1}$$

Let $R = R_* + \mu$. We seek the solution of equation

$$z' = B\ (R,\ z') \tag{4.2}$$

of the form

$$z' = z + \sum_{k=-1}^{1} \alpha_k \zeta_k, \qquad (z, \eta_k)_H = 0 \qquad (k = -1, 0, 1) \tag{4.3}$$

Substituting (4.3) into (4.2), we obtain

$$z - A(R_*)\, z = \mu \sum_{k=-1}^{1} \alpha_k A_0 \zeta_k + A(\mu)\, z + D\Big(z + \sum_{k=-1}^{1} \alpha_k \zeta_k,\ z + \sum_{k=-1}^{1} \alpha_k \zeta_k\Big) \equiv Qz \tag{4.4}$$

$$A_0(z) = (K_1(T\mathbf{r}), 0), \qquad z = (\mathbf{v}, T) \in H \tag{4.5}$$

We introduce the projective operator

$$\Pi\varphi = \varphi - \sum_{k=-1}^{1} \alpha_k \zeta_k, \quad \varphi \in H \tag{4.6}$$

and, using the solvability condition of Fredholm's equation, we write (4.4) in the equivalent form

$$z - A(R_*)\, z = \Pi Q z, \quad (Qz, \eta_k)_H = 0 \qquad (k = -1, 0, 1) \tag{4.7}$$

Before passing to finding small solutions of Eq. (4.7), we shall simplify the problem by using the theorem on branching of solutions of nonlinear equations with respect to transformations of any arbitrary compact group G [5]. In the problem here considered it is expedient to chose a group of three-dimensional space rotations for such group.

Let L_g be the representation of a group of rotations. Equation (4.4) is invariant with respect the L_g-transformations

$$L_g A(R)\, z = A(R)\, L_g z,\ \ L_g Q z = Q L_g z\ \ (z \in H, g \in \mathbf{G}) \tag{4.8}$$

In fact, the Laplace operator is invariant with respect to the L_g-transformation [10]. Let $g \in \mathbf{G}$ be an arbitrary rotation transforming the orthogonal coordinate system with its origin at the center of the sphere into system $y_l\ (i,\ l = 1,\ 2,\ 3)$. The equalities

$$u_i \frac{\partial u_k}{\partial x_i} = u_i \frac{\partial u_k}{\partial y_l} \frac{\partial y_l}{\partial x_i} = u_l \frac{\partial u_k}{\partial y_l},\ \ u_i \frac{\partial T}{\partial x_i} = u_i \frac{\partial T}{\partial y_l} \frac{\partial y_l}{\partial x_i} = u_l \frac{\partial T}{\partial y_l}$$

then become obvious. Hence

$$L_g(\mathbf{u}\nabla)\,\mathbf{u} = (\mathbf{u}_g\nabla)\,\mathbf{u}_g,\ \ L_g \mathbf{u}\nabla T = \mathbf{u}_g \nabla T_g\ \ (\mathbf{u}_g = L_g \mathbf{u},\ T_g = L_g T)$$

We note that the eigen-subspace Z_0 of operator $A\ (R_*)$ is invariant with respect to L_g. In [5] the representation L_g is called complete in Z_0, if for any pair $\zeta',\ \zeta'' \in Z_0$ we can indicate a $g \in G$ so that

$$L_g \zeta' = \alpha \zeta'' \qquad (\alpha > 0) \tag{4.9}$$

Lemma 4.1. The representation of the group of three-dimensional space rotations for $l = 1$ is complete in the eigen-subspace Z_0 of operator $A\ (R_*)$.

Proof. Let $\zeta' = \sum_{k=-1}^{1}\ _k\ \zeta_k$. The function of r is invariant at any arbitrary rotation, hence the analysis of $L_g\zeta'$ reduces to the analysis of (see (4.1))

$$L_g \sum_{k=-1}^{1} \alpha_k T^1_{mk}\,(^1\!/_2\pi - \varphi,\ \vartheta,\ 0)$$

Functions T^l_{mn} are simultaneously matrix elements of the L_g-representation [10]. The properties of functions T^l_{mn} $(^1\!/_2\,\pi - \varphi,\ \vartheta,\ 0)$ imply that

$$L_g \sum_{k=-1}^{1} \alpha_k T^1_{mk}(g_1) = \sum_{k=-1}^{1} \alpha_k T^1_{mk}(g_1 g) = \sum_{k=-1}^{1} \alpha_k \sum_{n=-1}^{1} T^1_{mn}(g_1)\, T^1_{nk}(g) =$$

$$= \sum_{n=-1}^{1} \beta_n T^1_{mn}(g_1), \quad \sum_{k=-1}^{1} \alpha_k T^1_{nk}(g) = \beta_n \quad (n = -1, 0, 1) \tag{4.10}$$

For $l = 1$ matrix T^l_{mn} is equivalent to the matrix of rotation $g\,(\varphi_1,\ \varphi_2,\ \vartheta)$ [10], where φ_1, φ_2 and ϑ are Euler's angles defining the rotation. From (4.9) and (4.10) now follows that the representation L_g will be complete for $l = 1$, if for any specified real α_k and β_n $(k,\ n = -1,\ 0,\ 1)$ such φ_1, φ_2 and ϑ can be found so as to satisfy equalities

$$\sum_{k=-1}^{1} \alpha_k g_{kn} = \alpha\beta_n \qquad (n = -1,\ 0,\ 1;\ \alpha > 0)$$

Simple calculations show that, for φ_2 determined from the equation $\beta_{-1}\cos\varphi_2 - \beta_0 \sin\varphi_2 = 0$ (if $\beta_{-1} = \beta_0 = 0$ then angle φ_2 is arbitrary) the angles φ_1 and ϑ are determined from equations

$$\alpha_{-1}\cos\varphi_1 + \alpha_0 \sin\varphi_1 = 0 \tag{4.11}$$

$$\alpha_1 - \alpha\,[\beta_1 \cos\vartheta + (\beta_0 \cos\varphi_2 + \beta_{-1}\sin\varphi_2)\sin\vartheta] = 0$$

where $\alpha > 0$ can be chosen so as to make Eq. (4.11) solvable. The Lemma is proved.

When $l = 1$ there is a subspace $H^\circ \subset H$ containing ζ_0 which consists of axisymmetric vectors orthogonal to ζ_{-1} and ζ_1 and invariant with respect to operators A and Q.

As shown in [5], small solutions originating at the bifurcation point R_* may be sought in the invariant subspace H°. All other small solutions branching off at that point are obtained from solutions derived by means of transformation L_g.

Small solutions of Eq. (4.7) in H° can be sought in the form of series expansions in powers of parameters α and μ (the zero subscript at α, ζ and η is henceforth omitted)

$$z = \sum_{p,q=0}^{\infty} z_{pq}\alpha^p\mu^q, \quad z_{00} = 0 \quad (z_{pq}, \eta_k)_H = 0 \quad (k = -1, 0, 1) \tag{4.12}$$

The coefficients appearing in (4.12) are defined by equations

$$z_{10} = z_{01} = z_{02} = z_{12} = z_{03} = 0,\ z_{20} - A\,(R_*)\,z_{20} = \Pi D\,(\zeta,\ \zeta) \tag{4.13}$$

and so forth. Substituting (4.12) together with the now known coefficients into the equation $(Qz,\ \eta)_H = 0$ (where $\alpha_{-1} = \alpha_1 = 0$ is assumed), we obtain the branching equations

$$\mu\alpha\,(A_0\zeta,\ \eta)_H + \alpha^2\,(D\,(\zeta,\ \zeta),\ \eta)_H + \alpha^3\,(D^\circ\,(z_{20},\ \zeta),\ \eta)_H + \ldots = 0$$

$$D^\circ\,(z_1,\ z_2) = D\,(z_1,\ z_2) + D\,(z_2,\ z_1) \tag{4.14}$$

Theorem 4.2. The emergence of new solutions of Eq. (4.2) takes place when the Rayleigh number passes through R_*, and one nonzero solution

$$z' = \gamma^{1/2}\mu^{1/2}\zeta + \gamma\mu z_{20} + O(\mu^{3/2}), \quad \gamma = -\frac{(A_0\zeta, \eta)_H}{(D^\circ(z_{20}, \zeta), \eta)_H} \tag{4.15}$$

corresponds to within an arbitrary rotation to every $R > R_*$ close to R_*.

Proof. The coefficient at α^2 in Eq. (4.15) vanishes by virtue of the known identities

$$(D(\zeta, \zeta), \eta)_H = -R_*^{-1}(K_1(\mathbf{u}\nabla)\mathbf{u}, \mathbf{u})_{H_1} - P(L_1(\mathbf{u}\nabla\theta), \theta)_{H_2} =$$

$$= -R_*^{-1}\int_\Omega (\mathbf{u}\nabla)\mathbf{u}\mathbf{u}\,dx - P\int_\Omega \theta\mathbf{u}\nabla\theta\,dx = 0$$

The coefficient at $\mu\alpha$ in the branching equation is of the form

$$(A_0\zeta, \eta)_H = \frac{1}{R_*^2}\|\mathbf{u}\|_{H_1}^2 \tag{4.16}$$

The coefficient at α^3 reduces to

$$\mathrm{I} \equiv (D^\circ(z_{20}, \zeta), \eta)_H = -R_*^{-1}\|\mathbf{v}_{20}\|_{H_1}^2 - \|T_{20}\|_{H_2}^2 + 2\int_\Omega r v_{20,r} T_{20}\,dx \tag{4.17}$$

It is not difficult to prove that this value is negative. For this it is sufficient to note that operator M defined by the equality $Mu = K_1(\mathbf{r}L_1(ru_r))$, which in H_1 is self-conjugate and positive [8]. The negativeness of I in (4.17) follows from the variational principle for operator M. Thus $\gamma > 0$ and as shown by Newton's diagram [17], the solution of Eq. (4.2) can be sought in the form of a series expansion in powers of $\varepsilon = \mu^{1/2}$

$$z' = \sum_{k=1}^{\infty} \varepsilon^k z_k$$

where z_k satisfies the equations

$$z_k - A(R_*)z_k = A_0 z_{k-2} + \sum_{m+n=k} D^\circ(z_m, z_n) \equiv \Psi_k \tag{4.18}$$

We seek z_k in the form

$$z_k = \delta_k\zeta + \delta_1^k\varphi_k, \qquad (\varphi_k, \eta)_H = 0$$

For $k = 1$ we have

$$z_1 = \delta_1\zeta, \quad \varphi_1 = 0$$

The condition of solvability of Eqs. (4.18) $(\psi_k, \eta)_H = 0$ for $k = 2, 3, \ldots$ we have

$$\delta_1^2 = -\frac{(A_0\zeta, \eta)_H}{(D^\circ(\varphi_2, \zeta), \eta)_H}, \qquad \delta_2 = 0$$

It is readily seen that $\delta_1^2 = \gamma$ and $\varphi_2 = z_{20}$. The Theorem is proved.

5. Calculation of convection. We apply to the integral equation (3.6) at $l = 1$ the iteration method proposed in [12] for nonnegative matrices. This method, which can be readily used for equations with strongly positive operators, consists in deriving consecutive two-sided estimates (3.9)

$$\min_{0\leqslant r\leqslant 1}\frac{\theta^{(n)}(r)}{\theta^{(n-1)}(r)} \leqslant \lambda_1 \leqslant \max_{0\leqslant r\leqslant 1}\frac{\theta^{(n)}(r)}{\theta^{(n-1)}(r)} = \|\theta^{(n-1)}\|_{\theta^{(n)}}, \quad \theta^{(n)} = G_1\theta^{(n-1)} \tag{5.1}$$

which converges to the first characteristic value λ_1 of operator G_1, while the sequence of functions $\theta^{(n)}$ converges to the eigenfunction θ_1. We take $u_0 \mid_{l=1} = r(1-r)$ from cone K_{u_0} as the first approximation, and write the condition of normalization as

$$\int_0^1 \theta_1(r)\, r^2 dr = 1$$

The fifth iteration carried out on a computer yielded estimate

$$8042.1 \leqslant R_* \leqslant 8042.3 \tag{5.2}$$

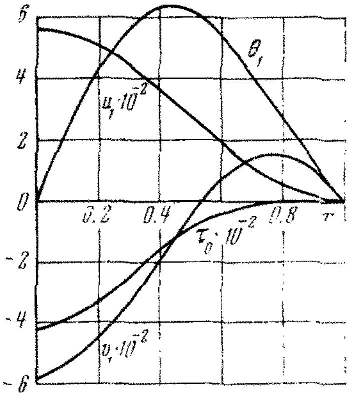

Fig. 1.

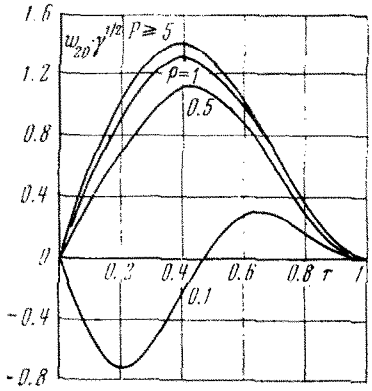

Fig. 2.

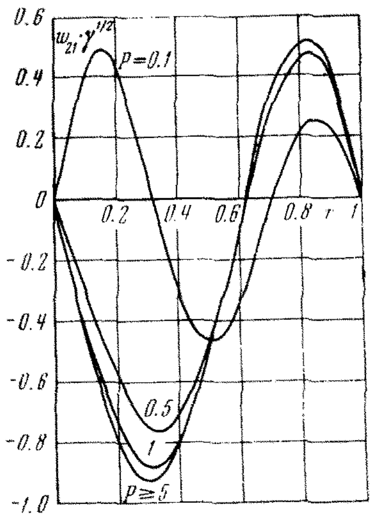

Fig. 3.

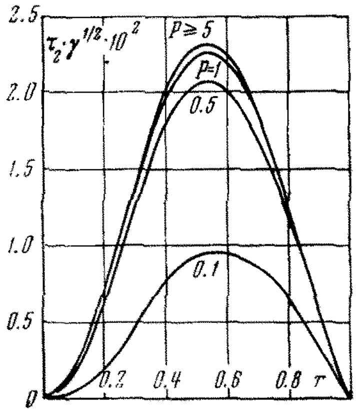

Fig. 4.

With three supporting functions the Ritx method had yielded $R_* = 8041.7$. Having determined R_* and θ_1, we find with the use of operator G_{21} from Eqs. (3.1) for $u_1(r)$ and $v_1(r)$ the expressions

$$ru_1 = R_* 2G_{21}\theta_1, \quad v_1 = -R_* \int_0^1 \left[\frac{\partial G_{21}(r, s)}{\partial r} + \frac{2}{r} G_{21}(r, s)\right] \theta_1(s)\, s^2 ds \tag{5.3}$$

Substituting in the expressions for $\theta(r, \vartheta)$ and $\mathbf{u}(r, \vartheta)$ the related generalized spherical functions, we find the explicit form of the eigenvector $\zeta = (\mathbf{u}, \theta)$ corresponding to R_*

$$u_r = u_1(r)\cos\vartheta,\ u_\varphi = 0,\ u_\vartheta = v_1(r)\sin\vartheta,\ \theta = \theta_1(r)\cos\vartheta \tag{5.4}$$

Vector $\mathbf{z}_{20} = (\mathrm{v}, T)$ is the solution of the last of Eqs. (4.13) or of the following boundary value problem:

$$\Delta \mathrm{v} = \nabla p + (\mathbf{u}\nabla)\mathbf{u} - R_*\mathbf{r}T,\ \nabla \mathrm{v} = 0$$
$$\Delta T = P\mathbf{u}\nabla\theta - rv_r,\ \mathrm{v}|_S = T|_S = 0,\ (\mathrm{v}, \mathbf{u})_{H_1} = 0,\ (T, \theta)_{H_2} = 0 \tag{5.5}$$

which is solvable by virtue of the identity

$$\int\limits_\Omega (\mathbf{u}\nabla)\mathbf{u}\mathbf{u}dx = 0, \qquad \int\limits_\Omega \theta\mathbf{u}\nabla\theta dx = 0$$

The solution of problem (5.5) is of the form

$$v_r = {}^1/_2\,(3\cos^2\vartheta - 1)\,w_{20}(r), \qquad v_\vartheta = 3\sin\vartheta\cos\vartheta\, w_{21}(r) \tag{5.6}$$
$$p = p_0(r) + {}^1/_2\,(3\cos^2\vartheta - 1)\,p_2(r),\ T = P\tau_0(r) + {}^1/_2\,(3\cos^2\vartheta - 1)\,\tau_2(r)$$

For p_0 and τ_0 we have

$$p_0(r) = \int\limits_0^r [R_* s\tau_0(s) - \Phi_1(s)]\,ds + \text{const}, \qquad \tau_0(r) = \int\limits_1^r \rho^{-2}\int\limits_0^\rho \Phi_4(s)\,s^2 ds d\rho \tag{5.7}$$

Here and in the following

$$\Phi_1(r) = {}^1/_3\left(u_1\frac{du_1}{dr} - \frac{2}{r}u_1v_1 - \frac{2}{r}v_1^2\right)$$
$$\Phi_2(r) = \frac{2}{3}\left(u_1\frac{du_1}{dr} + \frac{1}{r}u_1v_1 + \frac{1}{r}v_1^2\right),\ \ \Phi_3(r) = \frac{1}{3}\left(u_1\frac{dv_1}{dr} + \frac{1}{r}u_1v_1 + \frac{1}{r}v_1^2\right)$$
$$\Phi_4(r) = \frac{1}{3}\left(u_1\frac{d\theta_1}{dr} - \frac{2}{r}v_1\theta_1\right),\ \ \Phi_5(r) = \frac{2}{3}\left(u_1\frac{d\theta_1}{dr} + \frac{1}{r}v_1\theta_1\right)$$

Functions $w_{20}(r)$ and $\tau_2(r)$ are determined when solving the boundary value problem

$$D_2^2\omega_{20} = 6R_*\tau_2 + \Phi_2 + \frac{d}{dr}(\Phi_3),\ \omega_{20}'(0) < \infty,\ (rw_{20} = \omega_{20}) \tag{5.8}$$
$$-D_2\tau_2 = \omega_{20} - \Phi_5,\ \omega_{20}(0) = \omega_{20}(1) = \omega_{20}'(1) = \tau_2(1) = 0,\ \tau_2(0) < \infty$$

Problem (5.8) reduces to the integral equation with operator G_2 from (3.6)

$$\tau_2 = 6R_*G_2\tau_2 + \Phi,\quad \Phi = G_2\left(\Phi_2 + \frac{d}{dr}r\Phi_3\right) - G_{12}\Phi_5 \tag{5.9}$$

This equation was solved by the method of successive approximations, which is convergent, since the minimum characteristic value R_{12} of operator $6G_2$ is greater than R_*, as shown in Sect. 3. This characteristic value was derived by the iteration method and it was found that $R_* / R_{12} \approx 0.8$. If $\Phi(r)$ is taken as the zero-approximation, a satisfactory approximation is obtained at the 20 - 22nd iteration.

Having determined $\tau_2(r)$, we find $w_{20}(r)$ and $w_{21}(r)$

$$rw_{20} = 6R_*G_{22}\tau_2 + G_{22}\Psi \quad \left(\Psi = \Phi_2 + \frac{d}{dr}r\Phi_3\right)$$
$$w_{21}(r) = -\frac{1}{6}\int\limits_0^1\left[\frac{\partial G_{22}(r,s)}{\partial r} + \frac{2}{r}G_{22}(r,s)\right](6R_*\tau_2(s) + \Psi(s))\,s^2 ds \tag{5.10}$$

Constant γ is determined by expressions (4.5) - (4.7)

$$\gamma = -\|\mathbf{u}\|^2_{H_1}/R_*{}^2 I \tag{5.11}$$

Calculations were carried out on a computer. Functional dependence $\gamma(P)$ is fairly accurately defined by the formula $\gamma = 10^{-7}P^{-2}$ (a recalculation had shown a satisfactory correlation with the approximate results cited in [4]). The remaining results are shown in Figs. 1 - 4, while Fig. 5 shows the pattern of the convection flow appearing immediately after the loss of stability at $R = 8042.5$ and $P = 1$. This pattern does not appreciably vary with increasing Rayleigh number, although the motion intensity increases. Thus for $R = 8942$ and $P = 1$ the Reynolds number calculated from the maximum flow velocity is close to five.

6. Stability of convection motion. The stability of certain secondary flows was investigated in [6, 15] by the method of perturbations. Let us use this method for proving the stability of the convection motion of a fluid in a sphere for small ε

$$\mathbf{v}_0 = \gamma^{1/2}\varepsilon\mathbf{u} + \gamma\varepsilon^2\mathbf{v} + O(\varepsilon^3), \quad T_0 = T_{00} + \gamma^{1/2}\varepsilon\theta + \gamma\varepsilon^2 T + O(\varepsilon^3) \tag{6.1}$$

The investigation of stability reduces to the spectral problem with respect to parameter σ

$$-\sigma\mathbf{v}' + \Delta\mathbf{v}' = \nabla p' + (\mathbf{v}_0\nabla)\mathbf{v}' + (\mathbf{v}'\nabla)\mathbf{v}_0 - (R_* + \varepsilon^2)\mathbf{r}T', \quad \nabla\mathbf{v}' = 0$$

$$-\sigma PT' + \Delta T' = P\mathbf{v}_0\nabla T' + P\mathbf{v}'\nabla T_0, \quad \mathbf{v}'|_S = T'|_S = 0 \tag{6.2}$$

which is equivalent to the operator equation in space H

$$z' - A(R_*)z' = D^\circ(z_0, z') + \varepsilon^2 A_0 z' - \sigma C z' \tag{6.3}$$

where $z' = (\mathbf{v}', T')$, $z_0 = (\mathbf{v}_0, T_0)$ and $C = (K_1, PL_1)$. The basis in the eigensubspace Z_0 of operator $A(R_*)$ may be selected as follows:

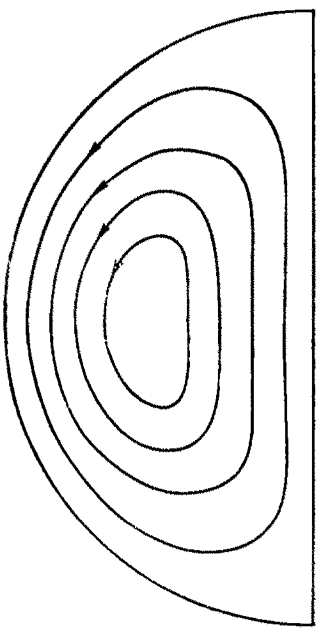

Fig. 5.

$$\zeta{:}\ u_r = u_1(r)\cos\vartheta, \quad u_\vartheta = v_1(r)\sin\vartheta,$$
$$u_\varphi = 0 \qquad \theta = \theta_1(r)\cos\vartheta$$
$$\zeta_1{:}\ u_r^{(1)} = u_1(r)\cos\vartheta\cos\varphi$$
$$u_\vartheta^{(1)} = v_1(r)\sin\vartheta\cos\varphi$$
$$u_\varphi^{(1)} = v_1(r)\cos\vartheta\sin\varphi$$
$$\theta^{(1)} = \theta_1(r)\cos\vartheta\cos\varphi \tag{6.4}$$
$$\zeta_{-1}{:}\ u_r^{(-1)} = u_1(r)\cos\vartheta\sin$$
$$u_\vartheta^{(-1)} = v_1(r)\sin\vartheta\sin\varphi$$
$$u_\varphi^{(-1)} = v_1(r)\cos\vartheta\cos\varphi$$
$$\theta^{(-1)} = \theta_1(r)\cos\vartheta\sin\varphi$$

We seek the solution of Eq. (6.3) in the form

$$z' = \sum_{k=-1}^{1}\delta_k\zeta_k + x, \quad (x, \eta_k)_H = 0$$
$$(k = -1, 0, 1) \tag{6.5}$$

Substituting (6.5) into (6.3) we obtain

$$x - A(R_*)x = \sum_{k=-1}^{1} \delta_k D^\circ(z_0, \zeta_k) + D^\circ(z_0, x) +$$
$$+ \varepsilon^2 \sum_{k=-1}^{1} \delta_k A_0 \zeta_k + \varepsilon^2 A_0 x - \sigma \sum_{k=-1}^{1} \delta_k C\zeta_k - \sigma Cx \equiv Q_1 x \tag{6.6}$$

or in the equivalent form

$$x - A\ (R_*)\ x = \Pi Q_1 x \tag{6.7}$$

where Π is the projecting operator (4.6).

It should be noted that the eigenvalues σ of Eq. (6.6) with a positive real part are uniformly bounded with respect to ε; $|\varepsilon| < \varepsilon_0$ [6, 18, 19]. Hence, in accordance with the theory of spectrum perturbation, only that eigenvalue σ_ε of Eq. (6.6) which originates from $\sigma_0 = 0$ can appear at small ε in the right-hand half-plane.

The x-solution of Eq. (6.7) can be sought in the form of a series expansion in powers of ε and σ

$$x = \sum_{p,\, q=0}^{\infty} \varepsilon^p \sigma^q x_{pq}, \qquad x_{00} = 0 \tag{6.8}$$

Vectors x_{pq} are defined by equations

$$x_{10} - A(R_*)x_{10} = \Pi \sum_{k=-1}^{1} \delta_k \gamma^{1/2} D^\circ(\zeta, \zeta_k)$$
$$x_{01} - A(R_*)x_{01} = -\Pi \sum_{k=-1}^{1} \delta_k C\zeta_k \tag{6.9}$$

and so on. Substituting series (6.8) in the right-hand side of Eq. (6.6), we obtain the conditions of solvability of this equation

$$\left(Q_1 \sum_{p,q=0}^{\infty} \varepsilon^p \sigma^q x_{pq},\ \eta_k\right)_H = 0 \quad (k = -1, 0, 1) \tag{6.10}$$

This yields three algebraic equations in ε, σ and δ_k. It is expedient to chose the following condition of normalization:

$$\sum_{k=-1}^{1} \delta_k^2 = 1$$

By modifying somewhat the Theorem 1 in [5] the number of equations in (6.10) can be reduced to two.

Let G' be a group of circle rotations ($g \in G'$ is a rotation about angle g). For $g \in G'$, $\mathbf{u} \in H_1$ and $T \in H_2$ we set

$$L_g'\mathbf{u}(r, \vartheta, \varphi) = \mathbf{u}(r, \vartheta, \varphi + g), \quad L_g'T(r, \vartheta, \varphi) = T(r, \vartheta, \varphi + g)$$

It is not difficult to verify that Eq. (6.3) and the eigen-subspace Z_0 of operator $A\ (R_*)$ are invariant with respect to operators L_g'. The representation of L_g' in space Z_0 has the following property: for any $\zeta' \in Z_0$ and $\zeta'' \in Z_0'$, where $Z_0' \subset Z_0$ spans ζ and ζ_1, it is possible with the use of (6.4) to find a g such that, $L_g'\zeta' = \zeta'' = \beta\zeta + \beta_1\zeta_1$.

There exists a subspace $H' \subset H$ which is invariant with respect to operators A, D and C, contains Z_0' consists of vectors $z = (\mathbf{v}, T)$ orthogonal to ζ_{-1} and satisfies the following conditions of evenness:

$$v_r(r,\vartheta,\varphi)=v_r(r,\vartheta,-\varphi),\qquad v_\vartheta(r,\vartheta,\varphi)=v_\vartheta(r,\vartheta,-\varphi)$$
$$v_\varphi(r,\vartheta,\varphi)=-v_\varphi(r,\vartheta,-\varphi),\qquad T(r,\vartheta,\varphi)=T(r,\vartheta,-\varphi)$$

This statement is directly confirmed by the system of Eqs. (6.2) equivalent to Eq. (6.3).

It can be readily shown, as was done in [5], that small solutions of Eq. (6.7) may be sought in subspace H'. If such solutions exist, all remaining small solutions in H are derived from these by the transformation L_g'.

Let us examine Eqs. (6.10) for $k=0$ and $k=1$. First, we note that (6.4) and (4.15) imply the equalities

$$(D^\circ(\zeta,\zeta_k),\eta_k)_H=0\qquad (k=0,1)$$
$$\gamma(D^\circ(z_{20},\zeta_k),\eta_k)_H+(A_0\zeta_k,\eta_k)_H=0\quad (k=0,1)$$

It follows from Eqs. (6.9) that

$$x_{pq}=\sum_{k=-1}^{1}\delta_k x_{pq}^{(k)}$$

where by now x_{pq}^k is independent of δ_k and for $k=0,1$ the system of Eqs. (6.10) becomes

$$\delta_k\{-\sigma(C\zeta_k,\eta_k)_H+\varepsilon^2\gamma^{1/2}(D^\circ(\zeta,x_{10}^{(k)}),\eta_k)_H+$$
$$+\varepsilon\sigma[\gamma^{1/2}(D^\circ(\zeta,x_{01}^{(k)}),\eta_k)_H-(Cx_{10}^{(k)},\eta_k)_H]+\ldots\}=0 \tag{6.11}$$

or

$$\delta_k F_k(\varepsilon,\sigma)=0,\qquad \delta^2+\delta_1^2=1\qquad (k=0,1) \tag{6.12}$$

Lemma 6.1. Functions $F(\varepsilon,\sigma)$ and $F_1(\varepsilon,\sigma)$ are linearly independent.

Proof. Equating the first of Eqs. (6.9) and the second of Eqs. (4.13) we obtain

$$x_{10}=2\gamma^{1/2}z_{20},\qquad z_{20}=(v_r,v_\vartheta,0;T) \tag{6.13}$$

$$x_{10}^{(1)}=2\gamma^{1/2}z_{20}^{(1)},\qquad z_{20}^{(1)}=(v_r\cos\vartheta,v_\vartheta\sin\varphi,v_\varphi^{(1)};T\cos\varphi) \tag{6.14}$$

For $k=0$ we then have (see (4.15))

$$\gamma^{1/2}(D^\circ(\zeta,x_{10}),\eta)_H=-2(A_0\zeta,\eta)_H \tag{6.15}$$

$$(C\zeta,\eta)_H=\frac{2\pi}{R_*}\int_0^\pi\int_0^1(u_r^2+u_\vartheta^2+PR_*\theta^2)\sin\vartheta\,d\vartheta\,r^2dr \tag{6.16}$$

Taking into consideration (6.4) and (6.14) for $k=1$, we have

$$\gamma^{1/2}(D^\circ(\zeta,x_{10}^{(1)}),\eta_1)_H=-(A_0\zeta,\eta)_H \tag{6.17}$$

$$(C\zeta_1,\eta_1)_H=\frac{\pi}{R_*}\int_0^\pi\int_0^1(u_r^2+u_\vartheta^2+u_\vartheta^2\operatorname{ctg}^2\vartheta+PR_*\theta^2)\sin\vartheta\,d\vartheta\,r^2dr \tag{6.18}$$

Thus, equating in Eqs. (6.11) for $k=0$ and $k=1$ the coefficients at σ and from (6.15) - (6.18) we obtain

$$\frac{(D^\circ(\zeta,x_{10}),\eta)_H}{(D^\circ(\zeta,x_{10}^{(1)}),\eta_1)_H}=2,\qquad \frac{(C\zeta,\eta)_H}{(C\zeta_1,\eta_1)_H}<2$$

The Lemma is proved.

It follows from Lemma 6.1 that system (6.12) splits into two independent equations

$$\delta_1 = 0,\ \delta = 1,\ F(\varepsilon, \sigma) = 0;\ \delta = 0,\ \delta_1 = 1,\ F_1(\varepsilon, \sigma) = 0 \qquad (6.19)$$

where $F_k(\varepsilon, \sigma)$ is an analytic function of ε and σ. As implied by equalities (6.16) and (6.18)

$$\partial F_k / \partial\sigma|_{\varepsilon=\sigma=0} = -(C\zeta_k, \eta_k)_H \neq 0 \qquad (k = 0, 1)$$

Hence by the implicit function theorem each of Eqs. (6.19) has a unique solution σ, which is analytic with respect to ε

$$\sigma_1 = -\varepsilon^2 \frac{2(A_0\zeta, \eta)_H}{(C\zeta, \eta)_H} + o(\varepsilon^2), \quad \sigma_2 = -\varepsilon^2 \frac{(A_0\zeta, \eta)_H}{(C\zeta_1, \eta_1)_H} + o(\varepsilon^2) \qquad (6.20)$$

It follows from (4.16), (6.16) and (6.18) that for small ε

$$\sigma_1 < 0, \qquad \sigma_2 < 0$$

Thus the convection flow defined by (6.1) is in the first approximation asymptotically stable, while according to [19] a nonlinear stability is also present. We note that the equilibrium state $T_{00} = C - {}^1/_2 r^2$ loses its stability, when R passes through R_*. To prove this it is sufficient to set in (6.11) $\gamma = 0$, which yields

$$\sigma = \varepsilon^2 \frac{(A_0\zeta_k, \eta_k)_H}{(C\zeta_k, \eta_k)_H} + o(\varepsilon^2) > 0 \qquad (k = 0, 1)$$

The authors thank V.I. Iudovich for the helpful discussion of this subject.

BIBLIOGRAPHY

1. Chandrasekhar, S., Hydrodynamic and Hydromagnetic stability. Clarendon Press, Oxford, 1961.
2. Roberts, P.H., Convection in a self-gravitating fluid sphere. J. Math., Vol. 12, pp.128-137, 1965.
3. Sherman, M., Toroidal and poloidal field representation for convective flow within s sphere. Phys. Fluids, Vol.11, №9, 1968.
4. Baldwin, P., Finite amplitude convection in a self-gravitating fluid sphere containing heat sources. Proc.Camb. Phil. Soc., Vol.63, №3, 1967.
5. Iudovich, V.I., Free convection and bifurcation. PMM Vol.31, №1, 1967.
6. Iudovich, V.I., Stability of convection flows. PMM Vol.31, №2, 1967.
7. Landau, L.D. and Lifshits, E.M., Mechanics of Continuous Media. 2nd ed., Gostekhteorizdat, 1954.
8. Ukhovskii, M.R. and Iudovich, V.I., On the equations of steady-state convection. PMM, vol.27, №2, 1963.
9. Iudovich, V.I., Example of the generation of secondary stationary or periodic flow when there is loss of stability of the laminar flow of a viscous incompressible fluid. PMM, Vol.29, №3, 1965.
10. Gel'fand, I.M., Minlos, R.A., and Shapiro Z.Ia., Representations of Group of Rotations and of Group of Lorentz and their Applications. Fizmatgiz, Moscow, 1958.
11. Babskii, V.G. and Sklovskaia, I.L., Hydrodynamics in weak force fields. Onset of stationary thermocapillary convection in a spherical layer of fluid in conditions of wightlessness. Izv. Akad. Nauk SSSR, MZhG, №3, 1969.

12. Gantmakher, F. R. and Krein, M. G., Oscillatory Matrices and Kernels and Small Vibrations of Mechanical Systems. Gostekhteorizdat, Moscow, 1950.
13. Ovchinnikova, S. N., Stability of Couette flow in the case of a wide gap between rotating cylinders. PMM, Vol. 34, №2, 1970.
14. Gantmakher, F. R., On the assymetric Kellog Kernels. Dokl. Akad. Nauk SSSR, Vol. 1(X), №1, 1936.
15. Krasnosel'skii, M. A., Positive Solutions of Operator Equations, Fizmatgiz, Moscow, 1962.
16. Bakhtin, I. A., On topological investigation methods of solutions of a particular class of nonlinear equations. Uspekhi matem. nauk, Vol. 20, №3, 1965.
17. Vainberg, M. M. and Trenogin, V. A., Theory of Branching of Solutions of Nonlinear Equations, "Nauka", Moscow, 1969.
18. Iudovich, V. I., Example of stability loss and onset of secondary flow of fluid in a closed vessel. Matem. sb., Vol. 74 (116), №4, 1967.
19. Iudovich, V. I., On the stability of stationary flows of viscous incompressible fluid. Dokl. Akad. Nauk SSSR, Vol. 161, №5, 1965.

Translated by J. J. D.

DIFFRACTION BY A CIRCULAR ISLAND OF LONG WAVES PRODUCED BY A RIPARIAN SOURCE

PMM Vol. 35, №6, pp. 1015-1022, 1971
S. Ia. SEKERZH-ZEN'KOVICH
(Moscow)
(Received April 6, 1971)

The diffraction of long cylindrical waves by a circular island situated in a rotating tank is considered. It is shown that, when the wavelength is small in comparison with the island radius, a resonanace capture of waves by the island takes place. Unlike in the author's paper [1] which analyzed the diffraction of monochromatic plane waves by a circular island in a rotating tank of constant depth, here the diffraction of cylindrical waves produced by a source at the island boundary is considered. As in [1], the wavelength is assumed to be considerable in comparison with the depth of the tank, but small relative to the island radius. A solution in the form of a conventional slowly convergent Fourier series is first derived, and then transformed by Watson's method into a fast convergent series, which makes it possible to determine the pattern of wave diffraction, at least along the island periphery. Many details of the derivation of solution have been omitted here. One of these details can be found in [1, 2], while others may be obtained by small alterations in the calculations presented in those papers.

1. Statement of problem. Fourier series for the elevation of fluid. A horizontally unbounded tank filled with a heavy perfect fluid rotates at angular velocity ω in a counterclockwise direction about a vertical axis. Depth of the tank is throughout uniform and equal h. The tank contains a source generating cylindrical